PHASE SEPARATION IN HETEROGENEOUS MEDIA

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Abstract. This paper is a review of recent results on a variational model in the context of the gradient theory for fluid-fluid phase transitions with small scale heterogeneities. We present a $\Gamma-$convergence result that identifies an anisotropic limiting surface energy, and investigate some of its properties.

1. Introduction

The study of pattern formation in equilibrium configurations phase separation is an extremely complex phenomenon which has attracted the interest of many mathematicians. In the case of homogeneous substances, variational models such as the Modica-Mortical functional (see [28, 29, 32]) and its vectorial (see [24, 4]), anisotropic (see [5, 23]), and non-isothermal variants (see [13]) have been proven capable of describing the stable configurations observed in experiments. For composite materials, it has been realized experimentally (see [6]) that the microscopic scale heterogeneities can affect the macroscopic equilibrium configurations as well as the dynamics of interfaces. Therefore, physics requires the mathematical models to include these microscopic effects.

In this paper, we consider a variational approach to the study of phase transitions in heterogeneous media in the case where the scale of the heterogeneities is the same as those at which the phase transitions phenomenon takes place. In particular, we study a Modica-Mortola like phase field model where the heterogeneities are modeled by oscillations in the potential. To be precise, let $d, N \geq 1$, fix an open bounded set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary and, for $\varepsilon > 0$, define the energy $\mathcal{F}_\varepsilon : H^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ as

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\varepsilon}, u(x)\right) + \varepsilon |\nabla u(x)|^2 \right] \mathrm{d}x\,. \tag{1}$$

Here $u \in H^1(\Omega;\mathbb{R}^d)$ represents the phase field variable. The assumptions that the double well potential $W : \mathbb{R}^N \times \mathbb{R}^d \to [0,\infty)$ has to satisfy differ according to the questions addressed, and therefore we will present them in each section.

We are interested in understanding what is the sharp interface limit as the parameter $\varepsilon \to 0$. Local minimizers of this limit under a mass constraint will describe equilibrium configurations.

Previous investigations on models related to the one considered in this paper have been undertaken by several authors. In particular, in [2] (see also [1]) Ansini, Braides and Chiadò Piat considered the case where oscillations are in the forcing term $f(\nabla u)$ (which generalizes $|\nabla u|^2$), while in [17] and [18] by Dirr, Lucia and Novaga investigated the interaction of the fluid with a periodic mean zero external field. Moreover, in [7], Braides and Zeppieri studied the Γ expansion of the scalar one dimensional case, allowing the zeros of the potential to jump in a specific way. Finally, the case of higher order derivatives is examined in [25] by Francfort and Müller.

2. Phase field model

In this section we present the results obtained in [11, 12, 9, 10].

2.1. **Sharp interface limit.** In order to study the sharp interface limit of the energy (1), we assume that the double well potential $W:\mathbb{R}^N\times\mathbb{R}^d\to[0,\infty)$ satisfies the following properties:

(A1) For all $p\in\mathbb{R}^d$, $x\mapsto W(x,p)$ is Q-periodic, where $Q:=(-1/2,1/2)^N$;
(A2) W is a Carathéodory function, *i.e.*,
  (i) for all $p\in\mathbb{R}^d$, the function $x\mapsto W(x,p)$ is measurable,
  (ii) for a.e. $x\in Q$, the function $p\mapsto W(x,p)$ is continuous;
(A3) There exist $z_1,z_2\in\mathbb{R}^d$ such that, for a.e. $x\in Q$, $W(x,p)=0$ if and only if $p\in\{z_1,z_2\}$,
(A4) There exists a continuous function $\widetilde{W}:\mathbb{R}^d\to[0,\infty)$, vanishing only at $p=z_1$ and at $p=z_2$, such that $\widetilde{W}(p)\le W(x,p)$ for a.e. $x\in Q$;
(A5) There exist $C>0$ and $q\ge 2$ such that

$$\frac{1}{C}|p|^q-C\le W(x,p)\le C(1+|p|^q)$$

for a.e. $x\in Q$ and all $p\in\mathbb{R}^d$.

Remark 2.1. The assumption (A2)(i) above is the strongest we can ask when modeling periodic inclusions of different materials. Indeed, when each cell Q is composed of k different inclusions of materials each in a region $E_1,\dots,E_k\subset Q$, the potential W takes the form

$$W(x,p):=\sum_{i=1}^k W_i(p)\chi_{E_i}(x)\,,$$

where $W_i:\mathbb{R}^d\to[0,\infty)$ are continuous functions with quadratic growth at infinity and such that $W_i(p)=0$ if and only if $p\in\{z_1,z_2\}$. Therefore the function W in the first variable is, in general, only measurable. Moreover, the continuity of W in the second variable, as well as the non degeneracy of the potential (A4) and the growth at infinity in the second variable (A5) are compatible with what is usually assumed in the physical literature.

The limiting functional will be an interfacial energy whose energy density is defined via a cell formula as follows.

Definition 2.2. For $\nu\in\mathbb{S}^{N-1}$, let $u_{0,\nu}:\mathbb{R}^N\to\mathbb{R}^d$ be the function

$$u_{0,\nu}(x):=\begin{cases} z_1 & \text{if } x\cdot\nu\le 0\,,\\ z_2 & \text{if } x\cdot\nu>0\,,\end{cases}$$

and denote by $\mathcal{Q}_\nu$ the family of cubes centered at the origin with unit length sides and having two faces orthogonal to ν. For $T>0$, $Q_\nu\in\mathcal{Q}_\nu$, and $\rho\in C^\infty_c(B(0,1))$ with $\int_{\mathbb{R}^N}\rho(x)\mathrm{d}x=1$, where $B(0,1)$ is the unit ball in $\mathbb{R}^N$, consider the class of functions

$$\mathcal{C}(\rho,Q_\nu,T):=\Big\{u\in H^1(TQ_\nu;\mathbb{R}^d)\,:\,u=u_{0,\nu}*\rho\text{ on }\partial(TQ_\nu)\Big\}\,.$$

We define the function $\sigma:\mathbb{S}^{N-1}\to[0,\infty)$ as

$$\sigma(\nu):=\lim_{T\to\infty}g(\nu,T)\,,$$

where, for each $\nu\in\mathbb{S}^{N-1}$ and $T>0$,

$$g(\nu,T):=\frac{1}{T^{N-1}}\inf\Big\{\int_{TQ_\nu}\big[W(y,u(y))+|\nabla u|^2\big]\,\mathrm{d}y\,:\,Q_\nu\in\mathcal{Q}_\nu,\,u\in\mathcal{C}(\rho,Q_\nu,T)\Big\}\,.$$

Remark 2.3. It was observed by Müller in [30] that, in the case the potential W is vectorial, in the definition of the cell formula it is not enough to take the minimum only on a single cell, but to consider the sequence of minima taken on larger and larger cells TQ_ν. In case the potential W is scalar, it is possible to reduce to a single cell problem with W replaced by W^{**} (see Lemma 4.1 and the remark after that, in [30]).

The main properties of the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ that are relevant for our study are collected in the following result. For the proof, see [11, Lemma 4.1, Remark 4.2, Lemma 4.3, Proposition 4.4].

Lemma 2.4. *The followings hold:*

(i) *For every $\nu \in \mathbb{S}^{N-1}$, the quantity $\sigma(\nu)$ is well defined and finite;*
(ii) *The value of $\sigma(\nu)$ does not depend on the choice of the mollifier ρ;*
(iii) *The map $\nu \mapsto \sigma(\nu)$ is upper semi-continuous on $\mathbb{S}^{N-1}$;*
(iv) *The infimum in the definition of $g(\nu, T)$ may be taken with respect to one fixed cube $Q_\nu \in \mathcal{Q}_\nu$. Namely, given $\nu \in \mathbb{S}^{N-1}$, for any $Q_\nu \in \mathcal{Q}_\nu$ it holds*

$$\sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\left\{ \int_{TQ_\nu} \left[W(y,u(y)) + |\nabla u|^2\right] \mathrm{d}y \,:\, u \in \mathcal{C}(\rho, Q_\nu, T) \right\}.$$

We are now in position to introduce the limiting functional.

Definition 2.5. Define the functional $\mathcal{F}_0 : L^1(\Omega;\mathbb{R}^d) \to [0,\infty]$ as

$$\mathcal{F}_0(u) := \begin{cases} \displaystyle\int_{\partial^* A} \sigma(\nu_A(x))\, \mathrm{d}\mathcal{H}^{N-1}(x) & \text{if } u \in BV(\Omega;\{z_1,z_2\}), \\ +\infty & \text{else,} \end{cases} \tag{2}$$

where $A := \{u = z_1\}$ and $\nu_A(x)$ denotes the measure theoretic external unit normal to the reduced boundary $\partial^* A$ of A at the point x.

Remark 2.6. Note that by Lemma 2.4(i), it holds $\mathcal{F}_0(u) < \infty$ for all $u \in BV(\Omega;\{z_1,z_2\})$, and, by Lemma 2.4(ii), the definition does not depend on the choice of the mollifier ρ.

Theorem 2.7. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,1)$ be a sequence such that $\varepsilon_n \to 0^+$ as $n \to \infty$. Assume that (A1), (A2), (A3), (A4), and (A5) hold.*

(i) *If $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\Omega;\mathbb{R}^d)$ is such that*

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\varepsilon_n}(u_n) < +\infty$$

then, up to a subsequence (not relabeled), $u_n \to u$ in $L^1(\Omega;\mathbb{R}^d)$, for some function $u \in BV(\Omega;\{z_1,z_2\})$.

(ii) *The functional $\mathcal{F}_0$ is the Γ-limit in the L^1 topology of the family of functionals $\{\mathcal{F}_{\varepsilon_n}\}_{n\in\mathbb{N}}$.*

Remark 2.8. The most interesting aspect of the above result is the anisotropic character of the limiting functional. This might come as a surprise since the initial functional $\mathcal{F}_\varepsilon$ is isotropic, but there is a hidden anisotropy: the possible mismatch between the directions of periodicity of W and the local orientation of the limiting interface $\partial^* A$ (see Figure 1).

We would like to comment on the main ideas behind the proof of Theorem 2.7. Compactness follows by using classical arguments (see [24]) since the non degeneracy assumption (A4) allows to reduce to the case of a non oscillating potential

$$\mathcal{F}_{\varepsilon_n}(u_n) \geq \int_\Omega \left[\frac{1}{\varepsilon_n}\widetilde{W}(u_n(x)) + \varepsilon_n |\nabla_n u(x)|^2 \right] \mathrm{d}x.$$

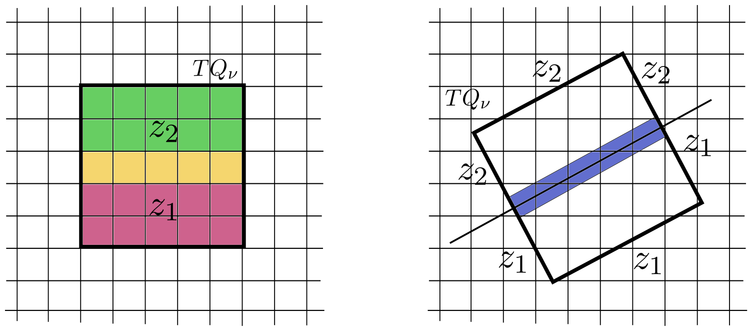

FIGURE 1. The source of anistropy for the limiting functional. If $\nu_A(x)$ is oriented with a direction of periodicity of W, the (local) recovery sequence would simply be obtained by using a rescaled version of the recovery sequence for $\sigma(\nu_A(x))$ in each yellow cube and by setting z_1 in the green region, and z_2 in the pink one. If, instead, $\nu_A(x)$ is not oriented with a direction of periodicity of W, the above procedure does not guarantee that we recover the desired energy, since the energy of such functions is *not* the sum of the energy of each cube.

The liminf inequality (see [11, Proposition 6.1]) is based on a standard blow-up argument (see [22]) at a point $x_0 \in \partial^* A$ to reduce to the case where the limiting function is $u_{0,\nu}$ and the domain is $Q_\nu \in \mathcal{Q}_\nu$, where $\nu = \nu_A(x_0)$. Then, a technical lemma (see [11, Lemma 3.1]) in the spirit of De Giorgi's slicing method (see [15]) allows to modify the given sequence $\{u_n\}_{n\in\mathbb{N}} \subset H^1(Q_\nu;\mathbb{R}^d)$ into a new sequence $\{v_n\}_{n\in\mathbb{N}} \subset H^1(Q_\nu;\mathbb{R}^d)$ with $v_n \to u_{0,\nu}$ in L^1, such that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n) \geq \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}(v_n),$$

and $v_n = \rho_n * u_{0,\nu}$ on ∂Q_ν , where $\rho_n(x) := \varepsilon_n^{-N}\rho(x/\varepsilon_n)$. The required inequality then follows by using a change of variable, and the definition of $\sigma(\nu)$ together with Lemma 2.4(iv).

The main challenges are related to the proof of the limsup inequality (see [11, Proposition 7.1]) for a function $u \in BV(\Omega, \{a,b\})$, which requires new geometric arguments. The idea is first to prove the result for functions $u \in BV(\Omega;\{a,b\})$ whose outer normals to the reduce boundary have rational coordinates, and then use the density of this class of functions in $BV(\Omega;\{a,b\})$ together with Reshetnyak's upper semi-continuity theorem (by Lemma 2.4(iii) the function $\nu \mapsto \sigma(\nu)$ is upper semi-continuous on $\mathbb{S}^{N-1}$) to conclude in the general case. In order to tackle the first step, we use a general strategy developed by De Giorgi, which can be seen as a sort of *reverse* blow-up argument: we consider the localized Γ-limsup as a map on Borel sets and we prove that it is indeed a Radon measure λ. This is done by using a simplification of the De Giorgi-Letta coincidence criterion for Borel measures (see [16]) by Dal Maso, Fonseca, and Leoni (see [14, Corollary 5.2]). Next, we show that λ is absolutely continuous with respect to the measure $\mu := \mathcal{H}^{N-1} \llcorner \partial^* A$. The result follows by proving that the density of λ with respect to μ at a point $x_0 \in \partial^* A$ is bounded above by $\sigma(\nu_A(x_0))$. It is in this step that we exploit the fact that $\nu_A(x_0) \in \mathbb{S}^{N-1} \cap \mathbb{Q}^{N-1}$: indeed, by using the fact that W is periodic (with a different period) also as a function on any cube Q whose faces are normal to directions in $\mathbb{S}^{N-1} \cap \mathbb{Q}^{N-1}$, we can estimate the energy of a configuration similar to that in Figure 1 on the left.

Remark 2.9. The strategy used to prove the above result is robust enough to be easily adapted to prove the analogous result when a mass constraint is enforced. Moreover, as a consequence of the Γ-limit result, we get that the function $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ is continuous, and its 1-homogeneous extension is convex.

The upshot of the foregoing result is that microscopic heterogeneities during phase transitions result in anisotropic surface tensions at the macroscopic level. Natural follow-up questions are:

(1) beyond convexity, what can one say about the effective surface tension σ? What functions σ are *attainable* as effective surface tensions of phase transitions in periodic media?
(2) considering the *gradient flow dynamics* of an energy as in (1), what are the $\varepsilon \to 0^+$ asymptotics ? Does one indeed obtain a suitable weak formulation of anisotropic mean curvature flow, by analogy with the isotropic setting?

In [9] we provide partial answers to the first question above, by relating it to a geometry problem. In [10], we address dynamics. In the rest of this survey we will summarize the results of [9], and a similar review of the results on dynamics will appear elsewhere [21].

In the sequel, we assume the product form of the potential W:

$$W(y,\xi) := a(y)(1-u^2)^2, \quad y \in \mathbb{R}^N, u \in \mathbb{R}. \tag{3}$$

Here $a : \mathbb{R}^N \to \mathbb{R}$ is Q-periodic, and non-degenerate in the sense that

$$\theta \leqslant a(y) \leqslant \Theta, \qquad y \in \mathbb{R}^N, \tag{4}$$

for some $0 < \theta < \Theta < \infty$. Note that assumptions (A1)-(A5) of Section 2.1 are satisfied with $z_1 = -1$, $z_2 = 1$ and $\widetilde{W} = W$. The fact that u is scalar-valued is crucial for a number of the results proven in [9, 10] since we use arguments based on the maximum principle. However, this isn't true of all the results, and we will indicate this as appropriate.

2.2. **Bounds on the Anisotropic Surface Tension σ.**

2.2.1. *A Geometric Framework.*

Consider the periodic Riemannian metric on $\mathbb{R}^N$ that is conformal to the Euclidean one, defined as follows: given points $x, y \in \mathbb{R}^N$, we set

$$d_{\sqrt{a}}(x,y) := \inf_{\gamma} \int_0^1 \sqrt{a(\gamma(t))}|\dot\gamma(t)|\,dt,$$

where the infimum is taken over Lipschitz continuous curves $\gamma : [0,1] \to \mathbb{R}^N$ such that $\gamma(0) = x, \gamma(1) = y$. It is easily seen that the formula defining $d_{\sqrt{a}}$ is independent of the parameterization of the competitor curves γ. Furthermore, standard arguments via the Hopf-Rinow theorem imply that $\mathbb{R}^N$ with the metric $d_{\sqrt{a}}$ is a complete metric space. Equivalently, geodesically complete: given any pair of points $x, y \in \mathbb{R}^N$ there exists a distance-minimizing geodesic joining them, whose length is equal to $d_{\sqrt{a}}(x,y)$ (see [32] for details). Now fix a direction $\nu \in \mathbb{S}^{N-1}$, and consider the plane Σ_ν through the origin with normal ν,

$$\Sigma_\nu := \{y \in \mathbb{R}^N : y \cdot \nu = 0\}.$$

Next, define the signed distance function in the $d_{\sqrt{a}}$−metric to the plane Σ_ν, via

$$h_\nu(y) := \operatorname{sgn}(y \cdot \nu) \inf_{z \in \Sigma_\nu} d_{\sqrt{a}}(y,z),$$

where the signum function is defined as

$$\operatorname{sgn}(t) := \begin{cases} 1 & t \geqslant 0, \\ -1 & t < 0. \end{cases}$$

It is easily shown (see [9, Lemma 2.2]) that h_ν is Lipschitz continuous, with

$$|\nabla h_\nu(y)| = \sqrt{a(y)} \qquad \text{at a.e. } y \in \mathbb{R}^N. \tag{5}$$

These observations, together with (4), yield

$$\begin{aligned} \sqrt{\theta}(y\cdot\nu) \leqslant h_\nu(y) \leqslant \sqrt{\Theta}(y\cdot\nu), \quad & y\cdot\nu \geqslant 0, \\ \sqrt{\Theta}(y\cdot\nu) \leqslant h_\nu(y) \leqslant \sqrt{\theta}(y\cdot\nu), \quad & y\cdot\nu < 0. \end{aligned} \tag{6}$$

In order to explain the relationship that the $d_{\sqrt{a}}$−metric bears with the anisotropic surface tension σ, it is useful to revisit the case $a \equiv 1$, and the celebrated Modica-Mortola example. In this case,

$$\sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf \left\{ \int_{TQ_\nu} \left[W(u(y)) + |\nabla u|^2\right] : u \in \mathcal{C}(\rho, Q_\nu, T) \right\}.$$

Elementary algebraic manipulations that effectively boil down to completing the square, yield that the infimum above is asymptotically reached by the one-dimensional profile satisfying *equipartition of energy.* This entails, in the model case of (3), that the optimal cost is achieved by the choice $u(y) = q\circ(y\cdot\nu)$, where $q := \tanh$. The associated cost is given by

$$\sigma(\nu) \equiv \sigma_0 := \int_{-\infty}^{\infty} \left[W(q\circ(y\cdot\nu)) + |\nabla(q\circ(y\cdot\nu))|^2\right] \, d(y\cdot\nu) = 2\int_{-1}^{1} \sqrt{W(s)}\, ds. \tag{7}$$

To make the connection to the $\sqrt{a}$− metric, we begin by noting that when $a \equiv 1$ we have $h_\nu(y) \equiv y\cdot\nu$. Our main motivation, then, is to obtain a similar formula that is exact when a is non-constant, or at least supplies reasonable bounds for the non-constant $\nu \mapsto \sigma(\nu)$. We do so by encoding the heterogeneous effects of a into the geometry of the underlying space, i.e., by working in the $\sqrt{a}$-metric. We turn to making these comments precise.

Fix $\nu \in \mathbb{S}^{N-1}$. Then, the cell formula defining $\sigma(\nu)$, proven in [11, 12] and specialized to our setting, reads (see Lemma 2.4 (iv))

$$\begin{aligned} \sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\Big\{ \int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right] \, dy : u \in H^1(TQ_\nu), \\ u = \rho * u_{0,\nu} \text{ on } \partial(TQ_\nu)\Big\}. \end{aligned}$$

Here, we recall that $u_{0,\nu}(y) := \operatorname{sgn}(y\cdot\nu)$ and ρ is any standard smooth normalized mollifier (it is shown in Lemma 2.4(ii) that $\sigma(\nu)$ is independent of this choice). A preliminary step is to observe, by De Giorgi's slicing method (see [9, Lemma A.1]) that, equivalently,

$$\begin{aligned} \sigma(\nu) = \lim_{T\to\infty} \frac{1}{T^{N-1}} \inf\Big\{ \int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right] \, dy : u \in H^1(TQ_\nu), \\ u = q\circ h_\nu \text{ along } \partial(TQ_\nu)\Big\}. \end{aligned} \tag{8}$$

For each fixed $T \gg 1$, by the Direct Method of the Calculus of Variations, the variational problem inside the limit has a minimizer. Such a minimizer is, perhaps, not unique, but for each T we select one, and call it u_T. We discuss various properties of u_T below in Section 2.2.2. In light of (8), it is clear by energy comparison, that

$$\sigma(\nu) \leqslant \liminf_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} [a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2]\,dy.$$

Towards proving the opposite bound, we introduce the function $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(z) := 2\int_0^z \sqrt{W(s)}\,ds.$$

This function plays a fundamental role in the Modica-Mortola analysis corresponding to $a \equiv 1$. For any $T \gg 1$, using (5) and completing squares, we find

$$\begin{aligned}
&\frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(u_T) + |\nabla u_T|^2\right]\,dy \\
&\quad= \frac{2}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \sqrt{W(u_T)}\nabla u_T\,dy + \frac{1}{T^{N-1}} \int_{TQ_\nu} \left|\nabla u_T - \sqrt{W(u_T)}\nabla h_\nu\right|^2 \\
&\quad\geq \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla(\phi(u_T))\,dy \\
&\quad= \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla(\phi(q\circ h_\nu))\,dy + \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla\left(\phi(u_T) - \phi(q\circ h_\nu)\right)\,dy \\
&\quad= \frac{1}{T^{N-1}} \int_{TQ_\nu} |\nabla h_\nu|^2 \phi'(q\circ h_\nu) q'(h_\nu)\,dy \qquad (9) \\
&\qquad+ \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla\left(\phi(u_T) - \phi(q\circ h_\nu)\right)\,dy \\
&\quad= \frac{1}{T^{N-1}} \int_{TQ_\nu} 2a(y)W(q\circ h_\nu)\,dy + \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla\left(\phi(u_T) - \phi(q\circ h_\nu)\right)\,dy \\
&\quad= \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[\, a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2\,\right]\,dy \\
&\qquad+ \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu \cdot \nabla\left(\phi(u_T) - \phi(q\circ h_\nu)\right)\,dy,
\end{aligned}$$

where in the last line we used the fact that the function $q\circ h_\nu$ achieves equipartition of energy. Indeed, by the definition of h_ν, we have

$$|\nabla(q\circ h_\nu)(y)|^2 = (q'(h_\nu(y))^2|\nabla h_\nu(y)|^2 = a(y)W(q(h_\nu(y))).$$

Defining

$$\begin{aligned}
\overline{\lambda}(\nu) &:= \limsup_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2\right]\,dy, \\
\underline{\lambda}(\nu) &:= \liminf_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2\right]\,dy,
\end{aligned}$$

provided we can control the error term

$$\limsup_{T\to\infty} \left| \frac{1}{T^{N-1}} \int_{TQ_\nu} \nabla h_\nu(y) \cdot \nabla\left(\phi(u_T) - \phi(q\circ h_\nu)\right)\,dy \right| := \lambda_0(\nu),$$

we observe that the test function $q\circ h_\nu$ gives two-sided bounds on $\sigma(\nu)$. Controlling the term λ_0 is complicated by the fact that it couples a product of weakly converging sequences (on expanding domains). Indeed, rescaling using $y = Tx$ in order to work in a fixed domain Q_ν, the two weakly converging factors making up the above product are:

(1) *the oscillatory factor:* by (5) and (4), the term $\{\nabla h_\nu(T\,\cdot)\}_T$, which is bounded in L^∞, converges weakly-*; and
(2) *the concentration factor:* the terms $\nabla\phi(u_T(T\,\cdot))$ and $\nabla\phi(q\circ h_\nu(T\,\cdot)$ converge weakly-* to measures (see Section 2.2.2 for precise statements).

In particular, as one of the factors converges to a measure, standard tools such as compensated compactness, used traditionally to pass to the limit in products of weakly converging sequences, are unavailable, and we must control this term "by hand". In Section 2.2.2 below, we obtain fine information on the concentration effects, in Section 2.2.3 we deduce partial results concerning the oscillatory effects. Finally, we put these together in Section 2.2.4 where we obtain bounds on $\lambda_0(\nu)$.

2.2.2. *Structure of Minimizers of the Cell Formula.* For fixed $T \gg 1$, let $u_T \in C^2(TQ_\nu)$ (by elliptic regularity) a minimizer of the energy

$$\int_{TQ_\nu} \left[a(y)W(u) + |\nabla u|^2\right]\, dy,$$

among competitors that equal $q\circ h_\nu$ along the boundary $\partial(TQ_\nu)$, and set

$$v_T(x) := u_T(Tx), \qquad x\in Q_\nu.$$

Lemma 2.10. *The functions v_T converge in L^1 to $u_{0,\nu} : Q_\nu \to \{\pm 1\}$.*

The proof of this lemma (see [9, Lemma 3.1]) is a nice application of the convexity of the one-homogeneous extension of σ (see Remark 2.9), using Jensen's inequality. The argument, without any changes, holds in the complete generality of the setting of [11] on the potential (vectorial, coupled, measurable dependence on the fast variable), and does not rely on the specific structure requested in (3). Combining Lemma 2.10 with the results of Caffarelli-Cordoba [8], we find that the level sets of v_T, for T sufficiently large, converge uniformly to $\Sigma_\nu \cap Q_\nu$.

Restricting ourselves to the scalar setting of (3), an argument using the strong maximum principle yields that for all $T < \infty$, we have

$$-1 < u_T(y) < 1,$$

(see [9, Lemma 3.2]) . In particular, $w_T := \frac{1}{\sqrt{2}}\tanh^{-1} u_T$ is well-defined, finite, and smooth in TQ_ν. Further, the function w_T verifies the elliptic boundary value problem

$$\begin{cases} \Delta w_T = \frac{4}{\sqrt{2}}\tanh w_T\big(|\nabla w_T|^2 - a(y)\big), & y\in TQ_\nu, \\ \\ w_T(y) = h_\nu(y) & y \in \partial(TQ_\nu). \end{cases}$$

Proposition 2.11. *Let w_T be as above, and let $T \gg 1$. There exist universal constants α_0 and $\eta_0 > 0$ such that the following holds:*

$$\begin{cases} \sqrt{\Theta}(y\cdot\nu) - \alpha_0 \geq w_T(y) \geq \sqrt{\theta}(y\cdot\nu) - \eta_0 & \text{if } w_T(y) > 0, \\ \\ -\sqrt{\theta}(y\cdot\nu) + \eta_0 \geq w_T(y) \geq -\sqrt{\Theta}(y\cdot\nu) + \alpha_0 & \text{if } w_T(y) < 0. \end{cases} \tag{10}$$

Proposition 2.11 asserts that, up to universal constants, the function w_T satisfies exactly the same growth rates as the function h_ν, see (6). To prove Proposition 2.11, consider, for instance, the lower bound in the first of the two inequalities in (10). The main observation is that the function $y \mapsto \zeta_T(y) := \frac{y\cdot\nu}{w_T(y)+\eta_0}$ satisfies an elliptic PDE that verifies a maximum principle. The remaining inequalities follow from similar arguments, and we refer the reader to [9, Proposition 3.4] for details.

2.2.3. *The Planar Metric Problem.* Our results on the distance function h_ν concern its large-scale behavior. The bounds on σ that we discuss in Section 2.2.4 below, depend solely on the large-scale behavior of the distance functions h_ν for which one can readily invoke efficient numerical algorithms, for example fast marching and sweeping methods [31].

A natural question concerns the large-scale homogenized behavior of h_ν, i.e., characterize the limit

$$\lim_{T\to\infty} \frac{h_\nu(Ty)}{T}, \qquad y \in \mathbb{R}^N,$$

in a suitable topology of functions. We fully resolve this question (see also [3]) by characterizing uniform limits of the function $h(T\cdot)/T$.

Theorem 2.12. *Let $\nu \in \mathbb{S}^{N-1}$. Then, there exists a real number $c(\nu) \in [\sqrt{\theta}, \sqrt{\Theta}]$, for each $K \subseteq \mathbb{R}^N$ compact, we have*

$$\lim_{T\to\infty} \sup_{y\in K} \left| \frac{1}{T} h_\nu(Ty) - c(\nu)(y\cdot\nu) \right| = 0.$$

Moreover, for all compact subsets L of $\mathbb{R}^N \setminus \Sigma_\nu$, we have

$$\lim_{T\to\infty} \sup_{y\in L} \left| \frac{1}{T(y\cdot\nu)} h_\nu(Ty) - c(\nu) \right| = 0.$$

We can interpret Theorem 2.12 as a homogenization result for the Eikonal equation in half-spaces. Indeed, it is well known (see for example [27]) that for each fixed $\nu \in \mathbb{S}^{N-1}$, the functions $k_m(y) := T_m^{-1} h_\nu(T_m(y))$ and $\ell(y) := c(\nu)(y\cdot\nu)$ are the unique viscosity solutions to

$$\begin{cases} |\nabla k_m| = \sqrt{a(T_m y)} & \text{in } \{y\cdot\nu \geq 0\}, \\ k_m = 0 & \text{on } \Sigma_\nu, \end{cases} \qquad \text{and} \qquad \begin{cases} |\nabla \ell| = c(\nu) & \text{in } \{y\cdot\nu \geq 0\}, \\ \ell = 0 & \text{on } \Sigma_\nu. \end{cases} \tag{11}$$

In fact, small modifications of our proofs permit us to prove almost periodic homogenization theorems for convex hamiltonians with Bohr almost periodic dependence on the fast variable, and Lipschitz continuous dependence on the slow variable (see [9, Theorem 1.4] for a precise statement). Theorem 2.12 shows that viscosity solutions of the PDEs on the left side of converge locally uniformly to the viscosity solution of the PDE on the right. A viscous and stochastic version of these equations (termed the "planar metric problem") was introduced by Armstrong and Cardaliaguet [3] and studied by others [19, 20] in the context of stochastic homogenization of geometric flows.

2.2.4. *Bounds on the Anisotropic Surface Tension.* As explained in the string of inequalities (9), the function $q \circ h_\nu$ provides tight upper and lower bounds for the effective anisotropy $\sigma(\nu)$. To be precise,

Theorem 2.13. *Let $\sigma : \mathbb{S}^{N-1} \to [0,\infty)$ be the anisotropic surface energy as in* (2.2). *Let $q : \mathbb{R} \to \mathbb{R}$ be defined by*

$$q(z) := \tanh(z), \quad z \in \mathbb{R}.$$

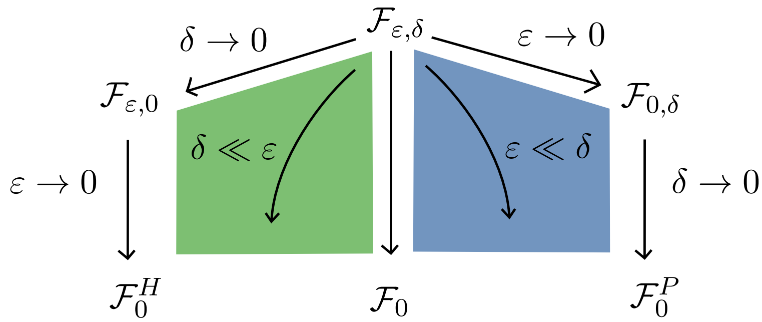

FIGURE 2. The situation when phase transitions and homogenization act at possibly different scales.

For $\nu \in \mathbb{S}^{N-1}$, *define*

$$\underline{\lambda}(\nu) := \liminf_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2\right] dy,$$

$$\overline{\lambda}(\nu) := \limsup_{T\to\infty} \frac{1}{T^{N-1}} \int_{TQ_\nu} \left[a(y)W(q\circ h_\nu) + |\nabla(q\circ h_\nu)|^2\right] dy.$$

There exist $\Lambda_0 > 0$ *and* $\lambda_0 : \mathbb{S}^{N-1} \to [0, \Lambda_0]$ *such that*

$$\overline{\lambda}(\nu) - \lambda_0(\nu) \leqslant \sigma(\nu) \leqslant \underline{\lambda}(\nu).$$

We do not expect these to agree when $\nu \in \mathbb{Q}^N \cap \mathbb{S}^{N-1}$ owing to finite-size effects: in such directions, h_ν is periodic, and the problem is restricted to an infinite strip, rather than all of space (see [9, Lemma 2.3]). However, generically, i.e., when ν is an irrational direction, we conjecture that $\lambda_0(\nu) = 0$, so that $\underline{\lambda}(\nu) = \overline{\lambda}(\nu)$.

2.3. **Open problems.** The studies presented above are a good source of interesting open problems. Here we list some of them.

2.3.1. *Different scales.* For $\varepsilon, \delta > 0$, consider the energy

$$\mathcal{F}_{\varepsilon,\delta}(u) := \int_\Omega \left[\frac{1}{\varepsilon} W\left(\frac{x}{\delta}, u(x)\right) + \varepsilon |\nabla u(x)|^2 \right] \mathrm{d}x.$$

defined for functions $u \in H^1(\Omega; \mathbb{R}^d)$. Here the parameter ε is related to the phase transition process, while δ describes the scale of periodicity. In the functional (1) we considered the case $\varepsilon = \delta$, namely when the two phenomena act at the same scale, but it is interesting to understand what happens when one scale is dominant with respect to the other. Heuristically, we expect the limiting energy to be the same in the green and in the blue region (see Figure 2). In particular, when $\varepsilon \ll \delta$ we expect the limiting functional $\mathcal{F}_0^P$ to be the homogenization of a surface energy functional, while in the other case, namely when $\delta \ll \varepsilon$, we expect to obtain the limit $\mathcal{F}_0^H$ of a classical Modica-Mortola functional whose potential is the homogenization of the original potential W.

This latter situation was investigated in [26] under the additional assumption that the positive infinitesimal sequences $\{\varepsilon_n\}_{n\in\mathbb{N}}$ and $\{\delta_n\}_{n\in\mathbb{N}}$ satisfy

$$\lim_{n\to\infty} \frac{\varepsilon_n^{3/2}}{\delta_n} = +\infty, \tag{12}$$

and by assuming the potential W to be locally Lipschitz in the second variable, uniformly in the first one. In particular, it was proved that the limiting functional is

$$\mathcal{F}_0^H(u) := \begin{cases} K_H \mathcal{P}(\{u = z_1\}; \Omega) & \text{if } u \in BV(\Omega; \{z_1, z_2\}), \\ \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mathcal{P}(\{u = z_1\}; \Omega)$ denotes the perimeter of the set $\{u = z_1\}$ in Ω, and the constant K_H is given by

$$K_H := 2 \inf \left\{ \int_0^1 \sqrt{W_H(\gamma(s))} |\gamma'(s)| \mathrm{d}s : \gamma \in C^1([0,1]; \mathbb{R}^d), \gamma(0) = z_1,\ \gamma(1) = z_2 \right\},$$

with the homogenized potential $W_H : \mathbb{R}^d \to [0, +\infty)$ given by $W_H(p) := \int_Q W(y, p)\, \mathrm{d}y$.

Some questions are still open: is this true also when $\delta \ll \varepsilon$ but without the extra assumption (12)? And what about the other regime?

2.3.2. *Sharpness of Bounds and Inverse Homogenization.* Various questions remain open from our discussion in 2.2. Our main contribution in that section was to relate the anisotropic surface tension σ to a purely geometric problem that had no concentration effects. Related to these bounds, we offer two open questions:

(1) Examine the tightness of the bounds in Theorem 2.13, and closely related,
(2) what does the set of effective anisotropies σ look like? In other words, which $\sigma : \mathbb{S}^{N-1} \to (0, \infty)$ with convex one-homogeneous extensions arise as a result of the homogenization procedure in [9]? Our bounds offer an approach to approximately solving this inverse homogenization question.

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